Algebra I : B.Sury
B.Math. (Hons.) Ist year
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Q 1.
(a) In a group G, suppose g is an element of order 143. Prove that there are uniquely defined elements $g_1, g_2 \in G$ with the properties

$$g = g_1 g_2 = g_2 g_1 \ , \ O(g_1) = 11, O(g_2) = 13.$$

(b) If N is a normal subgroup of a group G and $xy \in N$ for some $x, y \in G$, then show that $x^n y^n \in N$ for all natural numbers n.

Q 2.
Let $\theta : G \to G$ be an automorphism of a finite group G satisfying $\theta(g) = g$ if, and only if, $g = e$. Prove that every element of G must be of the form $x^{-1}\theta(x)$ for some $x \in G$. Further, if θ has order 2, show that $\theta(x) = x^{-1}$ for all x and that G is abelian.

Q 3.
Using Sylow's theorems or otherwise, prove that a group of order 33 must be cyclic.

Q 4.
Let G be a finite simple group and suppose n and $O(G)$ are relatively prime. Prove that each element of G is a product of n-th powers of elements of G.

Q 5.
Let G be a finite group and N be a normal subgroup such that $O(N)$ and $O(G/N)$ are relatively prime. If H is any subgroup of G such that $O(H)$ divides $O(N)$, show that H must be contained in N.

Q 6.
In a commutative ring with identity, prove that any maximal ideal is prime. Is this result true for the ring $2\mathbf{Z}$ of even integers ? Why ?

Q 7.
If $f_1, f_2, \cdots, f_n$ are in the ring $C([0,1], \mathbf{R})$ of real-valued continuous functions on $[0,1]$, show that they have a common zero x_0 in $[0,1]$ if the ideal generated by them is proper.

Q 8.
Let $\omega = e^{2i\pi/3}$, which is a cube root of unity. Consider the ring homomorphism $\theta : \mathbf{R}[X] \to \mathbf{C}$ which is the eveluation of any polynomial at ω. Prove that Ker θ is the principal ideal $(X^2 + X + 1)$.

Q 9.
If G is a finite group, p is a prime dividing $O(G)$ and R is the group ring $\mathbf{Z}_p[G]$ of G over $\mathbf{Z}_p$, prove that $(\sum_{g\in G} g)^2 = 0$ in R.